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A general worldline quantum inequality

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Abstract

Worldline quantum inequalities provide lower bounds on weighted averages of the renormalised energy density of a quantum field along the worldline of an observer. In the context of real, linear scalar field theory on an arbitrary globally hyperbolic spacetime, we establish a worldline quantum inequality on the normal ordered energy density, valid for arbitrary smooth timelike trajectories of the observer, arbitrary smooth compactly supported weight functions and arbitrary Hadamard quantum states. Normal ordering is performed relative to an arbitrary choice of Hadamard reference state. The inequality obtained generalises a previous result derived for static trajectories in a static spacetime. The underlying argument is straightforward and is made rigorous using the techniques of microlocal analysis. In particular, an important role is played by the characterisation of Hadamard states in terms of the microlocal spectral condition. We also give a compact form of our result for stationary trajectories in a stationary spacetimes.

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1 Introduction

One of the more surprising features of quantum field theory is that the renormalised energy density of a quantum field at a given spacetime point is unbounded from below as a function of the quantum state. However, this is not to say that the energy density can maintain an arbitrarily negative value for an arbitrary duration. Rather, it has been shown that there exist lower bounds—known as *quantum inequalities*—on weighted averages of the energy density taken either along the worldline of an observer [1, 2, 3, 4, 5] or over a spacetime region [6].

Restrictions of this type were first mooted by Ford [7] who argued that they are necessary to prevent macroscopic violations of the second law of thermodynamics. Subsequently, Ford and co-workers obtained explicit lower bounds on the worldline averages for a static observer in Minkowski space [1] or more generally in static spacetimes [2], for the case where the weight is the Lorentzian function $f_\tau(t) = t/(\pi(t^2 + \tau^2))$. In the same vein, but by different methods, the present author has found similar bounds valid for arbitrary smooth positive weights, with Eveson [4] (in Minkowski space) and Teo [5] (for static spacetimes). For the specific case of a massless field in two-dimensional Minkowski space, Flanagan [3] had previously found a bound valid for arbitrary weight which, moreover, he showed to be optimal. (It is in fact tighter than the corresponding bound of [4] by a factor of $3/2$.) Helfer [6] has recently given a rigorous proof that certain spacetime averages of the stress energy tensor are bounded below, but his approach has not as yet led to explicit formulae of the type obtained in the worldline case.

In this paper we establish a rigorous and general worldline quantum inequality, which generalises and makes precise the results of [4, 5]. Consider a real, minimally coupled Klein–Gordon field with mass $m \geq 0$ on an N -dimensional ($N \geq 2$)¹ globally hyperbolic Lorentzian spacetime (M, \mathbf{g}) , and let ω_0 be a fixed (globally) Hadamard state on the usual algebra $\mathfrak{A}(M, \mathbf{g})$ of smeared fields on (M, \mathbf{g}) . Let γ be any timelike curve in M , parametrised by proper time. For any other globally Hadamard state ω on $\mathfrak{A}(M, \mathbf{g})$, let

$$\rho_\omega(\tau) = \langle u^a(\tau)u^b(\tau) : T_{ab}(\gamma(\tau)) : \rangle_\omega \quad (1.1)$$

be the expected normal ordered energy density in state ω observed along γ , where we normal order with respect to ω_0 . Then, as we will show, the quantum inequality

$$\inf_\omega \int d\tau |g(\tau)|^2 \rho_\omega(\tau) \geq - \int_0^\infty \frac{d\alpha}{\pi} \int d\tau d\tau' \overline{g(\tau)} g(\tau') e^{-i\alpha(\tau-\tau')} \langle T \rangle_{\omega_0}(\tau, \tau') \quad (1.2)$$

holds for each smooth compactly supported (complex-valued) $g \in C_0^\infty(\mathbb{R})$, where the infimum is taken over all globally Hadamard states on $\mathfrak{A}(M, \mathbf{g})$ and $\langle T \rangle_{\omega_0}$ is the expectation in state ω_0 of the unrenormalised energy density, point-split along γ .² This

¹There are well-known pathologies associated with massless two-dimensional fields (see, e.g., [8]) which strictly require a separate treatment. We will not do this in the present paper, but expect our results to hold in this case because they are proved by local methods and in any case concern only derivatives of the field.

²See Sect. 3 for the precise definition. The quantity $\langle T \rangle_{\omega_0}$ is a bi-distribution on \mathbb{R}^2 ; we use the integral notation for ease of presentation.

‘difference quantum inequality’ immediately implies a bound on the renormalised (rather than normal ordered) energy density since $\rho_\omega = \rho_\omega^{\text{ren}} - \rho_{\omega_0}^{\text{ren}}$. We emphasise that this is not expected to be the best possible worldline bound. Indeed, (modulo the caveat of footnote 1) our bound reduces to the result of [4] for massless fields in two dimensional Minkowski space and is therefore strictly weaker than that of [3] in this case. In addition, the definition of $\langle T \rangle_{\omega_0}$ involves a choice of orthonormal frame along γ ; it is currently unclear whether some choices give a tighter bound than others.

The proof of (1.2) is extremely simple in outline. The energy density ρ_ω is equal to the restriction to the diagonal $\tau' = \tau$ of the smooth point-split normal ordered energy density defined by

$$\langle : T : \rangle_\omega(\tau, \tau') = \langle T \rangle_\omega(\tau, \tau') - \langle T \rangle_{\omega_0}(\tau, \tau'). \quad (1.3)$$

It follows that

$$\int d\tau |g(\tau)|^2 \rho_\omega(\tau) = \int_0^\infty \frac{d\alpha}{\pi} \int d\tau d\tau' \overline{g(\tau)} g(\tau') e^{-i\alpha(\tau-\tau')} \langle : T : \rangle_\omega(\tau, \tau'), \quad (1.4)$$

where the restriction to $(0, \infty)$ is possible because $\langle : T : \rangle_\omega$ is real and symmetric in τ, τ' . On the other hand, it follows from positivity of ω that the (unrenormalised) quantity $\langle T \rangle_\omega$ is a distribution of *positive type*, that is,

$$\int d\tau d\tau' \overline{f(\tau)} f(\tau') \langle T \rangle_\omega(\tau, \tau') \geq 0 \quad (1.5)$$

for all $f \in C_0^\infty(\mathbb{R})$. The required result is now obtained by substituting (1.3) in (1.4) and applying the inequality (1.5) in the case $f(\tau) = e^{-i\alpha\tau} g(\tau)$.

Sects. 2–4 will be concerned with making this argument properly rigorous. The key points are: the definition of $\langle T \rangle_\omega$ as a distribution on \mathbb{R}^2 ; the proof that it is of positive type; and, most important of all, the proof that the right-hand side of (1.2) converges (without which the result would be trivial). The main techniques employed are drawn from microlocal analysis [9] and in particular the characterisation of globally Hadamard states in terms of the *microlocal spectral condition* [10, 11, 12]. However, the flavour of the argument is easily given with reference to the case of a static trajectory $\gamma(\tau) = (\tau/|g_{tt}|^{1/2}, x_0)$ in a static spacetime (M, \mathbf{g}) , using the static ground state as the reference state ω_0 . If we express the quantum field as a sum (or integral) of mode functions

$$\varphi(t, x) = \sum_\lambda e^{-i\omega_\lambda t} U_\lambda(x) a_\lambda + e^{i\omega_\lambda t} \overline{U_\lambda(x)} a_\lambda^\dagger \quad (1.6)$$

with $[a_\lambda, a_{\lambda'}^\dagger] = \delta_{\lambda\lambda'} \mathbb{1}$, the point-split energy density $\langle T \rangle_{\omega_0}$ turns out to be

$$\langle T \rangle_{\omega_0}(\tau, \tau') = \sum_\lambda C_\lambda e^{-i\omega_\lambda(\tau-\tau')} |g_{tt}|^{-1/2} \quad (1.7)$$

where

$$C_\lambda = \frac{1}{2} \left[\left(\frac{\omega_\lambda^2}{|g_{tt}|} + m^2 \right) |U_\lambda(x_0)|^2 + \nabla_i U_\lambda|_{x_0} \nabla^i \overline{U}_\lambda|_{x_0} \right] \quad (1.8)$$

for this trajectory.³ The inner integral in the right-hand side of inequality (1.2) is $\sum_\lambda C_\lambda |\widehat{g}(\alpha + \omega_\lambda |g_{tt}|^{-1/2})|^2$, where \widehat{g} is the Fourier transform of g (see Sect. 2 for our conventions). Provided the ω_λ are bounded from below and have reasonable asymptotic behaviour along with the C_λ 's, this quantity will converge for each fixed α due to the rapid decay of $\widehat{g}(\omega)$ as $\omega \rightarrow \infty$, and will itself decay rapidly as $\alpha \rightarrow \infty$. Accordingly the right-hand side of (1.2) converges for arbitrary $g \in C_0^\infty(\mathbb{R})$ and thus constitutes a non-trivial bound:

$$\begin{aligned} \inf_\omega \int d\tau |g(\tau)|^2 \rho_\omega(\tau) &\geq - \int_0^\infty \frac{d\alpha}{\pi} \sum_\lambda C_\lambda |\widehat{g}(\alpha + \omega_\lambda |g_{tt}|^{-1/2})|^2 \\ &= - \frac{1}{\pi} \int_0^\infty du |\widehat{g}(u)|^2 \sum_{\substack{\lambda \text{ s.t.} \\ \omega_\lambda \leq |g_{tt}|^{1/2} u}} C_\lambda, \end{aligned} \quad (1.9)$$

which (modulo a change in parametrisation) reproduces the result of [5] (and hence that of [4] in the Minkowskian case). However, this argument would clearly require some delicate consideration of the asymptotics of the mode functions and their energies before it could be made rigorous. Our analysis completely circumvents this problem provided we can assume that the static ground state ω_0 is Hadamard—as is true for a wide class of static spacetimes [13]—because the microlocal spectral condition [10] then applies and entails almost immediately that the inner integral in (1.2) decays rapidly as $\alpha \rightarrow +\infty$. Moreover, this approach also applies in the general globally hyperbolic case.

In Sect. 5 we show how the general form of (1.9) persists in the case of a stationary trajectory γ in a stationary spacetimes, with ω_0 chosen to be a stationary ground state. More precisely, we will show that inequality (1.2) becomes

$$\inf_\omega \int d\tau |g(\tau)|^2 \rho_\omega(\tau) \geq - \frac{1}{\pi} \int_0^\infty du |\widehat{g}(u)|^2 Q(u) \quad \forall g \in C_0^\infty(\mathbb{R}), \quad (1.10)$$

where $Q(u)$ is a polynomially bounded function whose growth is expected (on dimensional grounds) to be $O(u^N)$ as $u \rightarrow +\infty$. This bound may be reformulated as the assertion that, for each globally Hadamard ω , the operator $H_\omega = Q(|iD|) + \rho_\omega$ is positive on $L^2(\mathbb{R})$ where D denotes differentiation on \mathbb{R} and ρ_ω acts by multiplication. This general viewpoint has recently been explored by Teo and the present author [14] for massless fields in even dimensional Minkowski space in relation to the *quantum interest conjecture* of Ford & Roman [15]. Indeed, a similar reformulation can be made even in the general globally hyperbolic case, and the resulting pseudodifferential operator may well repay investigation. We conclude in Sect. 6 with a brief summary and outlook.

³The index i runs over spatial coordinates and is raised and lowered using the positive definite spatial metric.

2 Preliminaries

2.1 Algebraic quantum field theory

We will work in the framework of algebraic quantum field theory (see [16] for a review). Suppose (M, \mathbf{g}) is an N -dimensional globally hyperbolic Lorentzian manifold ($N \geq 2$) with signature $+-\cdots-$. The classical Klein–Gordon equation on (M, \mathbf{g}) is

$$(\square_{\mathbf{g}} + m^2)\phi = 0, \quad (2.1)$$

where $m \geq 0$ and $\square_{\mathbf{g}} = g^{ab}\nabla_a\nabla_b$. Here, ∇_a is the derivative operator compatible with \mathbf{g} and Latin indices are to be understood as abstract tensor indices [17].

The theory is quantised by introducing an algebra $\mathfrak{A}(M, \mathbf{g})$ of observables on (M, \mathbf{g}) . To do this, the set of smooth compactly supported complex-valued test functions $C_0^\infty(M)$ is first used to label a set of abstract objects $\{\phi(f) \mid f \in C_0^\infty(M)\}$ (interpreted as smeared fields) which generate a free unital $*$ -algebra \mathfrak{A} over \mathbb{C} . The algebra $\mathfrak{A}(M, \mathbf{g})$ is defined to be the quotient of \mathfrak{A} by the relations⁴

$$(Q1) \text{ Hermiticity: } (\phi(f))^* = \phi(\bar{f}) \text{ for all } f \in C_0^\infty(M)$$

$$(Q2) \text{ Linearity: } \phi(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 \phi(f_1) + \lambda_2 \phi(f_2) \text{ for all } \lambda_i \in \mathbb{C}, f_i \in C_0^\infty(M)$$

$$(Q3) \text{ Field Equation: } \phi((\square_{\mathbf{g}} + m^2)f) = 0 \text{ for all } f \in C_0^\infty(M).$$

$$(Q4) \text{ CCR's: } [\phi(f_1), \phi(f_2)] = i\Delta_{\mathbf{g}}(f_1 \otimes f_2)\mathbb{1} \text{ for all } f_i \in C_0^\infty(M).$$

Here, $\Delta_{\mathbf{g}} = \Delta_{\mathbf{g}}^A - \Delta_{\mathbf{g}}^R$ is the advanced-minus-retarded fundamental bisolution corresponding to the Klein–Gordon operator $\square_{\mathbf{g}} + m^2$. The consistency of relation (Q4) is, of course, a consequence of the fact that $\Delta_{\mathbf{g}}$ is anti-symmetric (i.e., $\Delta_{\mathbf{g}}(f_1 \otimes f_2) = -\Delta_{\mathbf{g}}(f_2 \otimes f_1)$) and real (i.e., $\Delta_{\mathbf{g}}(\bar{f}_1 \otimes \bar{f}_2) = \overline{\Delta_{\mathbf{g}}(f_1 \otimes f_2)}$). Thus $\mathfrak{A}(M, \mathbf{g})$ consists of complex polynomials in the $\phi(f)$, their adjoints $\phi(f)^*$ and the identity $\mathbb{1}$ with the rule that any two such polynomials are equivalent if one may be manipulated into the other using the above rules.

A state on $\mathfrak{A}(M, \mathbf{g})$ is linear functional $\omega : \mathfrak{A}(M, \mathbf{g}) \rightarrow \mathbb{C}$ which is normalised so that $\omega(\mathbb{1}) = 1$ and positive in the sense that $\omega(A^*A) \geq 0$ for all $A \in \mathfrak{A}(M, \mathbf{g})$. The two-point function of a state is the bilinear functional on $C_0^\infty(M) \otimes C_0^\infty(M)$ given by

$$\omega_2(f \otimes g) = \omega(\phi(f)\phi(g)). \quad (2.2)$$

Throughout this paper, we will restrict to states whose two-point function is a bi-distribution. It is an immediate consequence of positivity and the hermiticity relation

⁴We will follow Radzikowski's conventions [10] for these axioms and for the definition of Fourier transformation used below. Different conventions are used, for example, in [12] which leads to some differences in the appearance of certain expressions.

(Q1) that the two-point function is a distribution of *positive type*, i.e., $\omega_2(\bar{f} \otimes f) \geq 0$ for all $f \in C_0^\infty(M)$. More generally, hermiticity also implies that $\omega_2(f \otimes g) = \omega_2(\bar{g} \otimes \bar{f})$, and this together with the formula

$$\omega_2(f \otimes g) = \frac{1}{2}\omega_2(f \otimes g + g \otimes f) + \frac{i}{2}\Delta_g(f \otimes g)\mathbb{1} \quad (2.3)$$

and the properties of Δ_g shows that, as is well known, all two-point functions have a common anti-symmetric part, and have real symmetric parts.

In order to consider the stress-energy tensor of the field, we must further restrict attention to the class of globally Hadamard states, for which the renormalised stress-energy tensor may be defined by the usual point-splitting method. In [18], Kay and Wald gave a rigorous definition of a globally Hadamard quasifree state⁵ in terms of the Hadamard series. The work of Radzikowski [10], subsequently modified by other authors [11, 12], has led to a reformulation of this condition in terms of microlocal analysis, which we now briefly review.

2.2 Microlocal analysis and the Hadamard condition

Microlocal analysis is a powerful technique for analysing the singularity structure of distributions. Define the Fourier transform on \mathbb{R}^n by

$$\hat{u}(k) = \int d^n x e^{ik \cdot x} u(x). \quad (2.4)$$

It is well known that a function $u \in C_0^\infty(\mathbb{R}^n)$ has the property that its Fourier transform $\hat{u}(k)$ decays faster polynomially as k tends to infinity in any direction. This will not be true in general if u is a distribution of compact support, for which we define the Fourier transform by $\hat{u}(k) = u(e_k)$, where $e_k(x) = e^{ik \cdot x}$. With this in mind, we define $\Sigma(u)$ to be the set of all $k \in \mathbb{R}^n \setminus \{0\}$ which have no conical⁶ neighbourhood V in which \hat{u} is of rapid decrease. To be precise, \hat{u} is said to decrease rapidly in the cone V if, for each $N = 1, 2, \dots$, there exists a constant C_N such that

$$|\hat{u}(k)| \leq C_N(1 + |k|)^{-N} \quad \forall k \in V, \quad (2.5)$$

where $|k|$ denotes the Euclidean norm of k . The set $\Sigma(u)$ thus describes the ‘singular directions’ of u .

The wave front set provides more detailed information about the singularities of u by localising the idea of a singular direction. If $u \in \mathcal{D}'(X)$ for some open $X \subset \mathbb{R}^n$ and $x \in X$, we define $\Sigma_x(u) = \cap_\varphi \Sigma(\varphi u)$ where the intersection is taken over all smooth compactly

⁵A quasifree state has a vanishing one-point function and n -point functions defined recursively in terms of the two-point function for $n > 2$. See [18] for the full definition.

⁶A cone in \mathbb{R}^n is a subset V such that $k \in V$ implies $\lambda k \in V$ for all $\lambda > 0$.

supported functions $\varphi \in C_0^\infty(X)$ which are nonzero at x . We may now define the *wave front set* $\text{WF}(u)$ of u as follows:

$$\text{WF}(u) = \{(x, k) \in X \times \mathbb{R}^n \setminus \{0\} \mid k \in \Sigma_x(u)\}. \quad (2.6)$$

It is immediate from this definition that $\text{WF}(Pu) \subset \text{WF}(u)$ for any partial differential operator P (of finite order and with smooth coefficients) on X . The non-expansion of the wave front set under such operators makes it a natural and useful tool for analysing solutions to PDEs.

The wave front set may be lifted from distributions on open sets of \mathbb{R}^n to distributions on a smooth manifold X . In treating distributions on a manifold, we will always assume the existence of a smooth positive density σ_X which identifies each smooth $F \in C^\infty(X)$ with a distribution in $'(X)$ by the formula $F(f) = \int_X \sigma_X(x) F(x) f(x) dx$. In particular, when treating a spacetime (M, \mathbf{g}) we will always use $|\det \mathbf{g}|^{1/2}$ as this density. The wave front set is defined as follows: if (X_κ, κ) is a chart in X and $u \in '(X)$ then the restriction to X_κ of $\text{WF}(u)$ is the subset of $\dot{T}^*(X)$ given by $\kappa^* \text{WF}(u \circ \kappa^{-1})$, where $\dot{T}^*(X)$ denotes the cotangent bundle of X less its zero section, and

$$\kappa^* \text{WF}(u \circ \kappa) = \{(x, {}^t \kappa'(x) \xi) \mid (\kappa(x), \xi) \in \text{WF}(u \circ \kappa^{-1})\}. \quad (2.7)$$

This definition is in fact invariant under changes of coordinate, and an intrinsic definition may be given [19]. The property $\text{WF}(Pu) \subset \text{WF}(u)$ continues to hold for partial differential operators on X .

Radzikowski showed how the (global) Hadamard condition of [18] could be rephrased in this language. A two-point function on a globally hyperbolic manifold (M, \mathbf{g}) will be said to obey the microlocal spectral condition if

$$\text{WF}(\omega_2) = \left\{ (x, k; x', -k') \in \dot{T}^*(M \times M) \mid (x, k) \sim (x', k') \text{ and } k \in \overline{V}_+ \right\}, \quad (2.8)$$

where \overline{V}_+ is the closed forward cone $g^{\mu\nu} k_\mu k_\nu \geq 0, k_0 \geq 0$, and $(x, k) \sim (x', k')$ if and only if x and x' are null related with k cotangent to the connecting null geodesic at x and k' its parallel transport to x' with the convention that $(x, k) \sim (x, k')$ if and only if $k = k'$ is null. Radzikowski [10] proved that, in four dimensions, a two-point function of a state on $\mathfrak{A}(M, \mathbf{g})$ obeys the global Hadamard condition of [18] if and only if it obeys (2.8).⁷

For our purposes, the microlocal spectral condition is much more convenient than the Hadamard series definition. We shall therefore say that, for a globally hyperbolic spacetime (M, \mathbf{g}) of arbitrary spacetime dimension $N \geq 2$, a state on $\mathfrak{A}(M, \mathbf{g})$ has a globally Hadamard two-point function if and only if it is a bi-distribution which obeys the microlocal spectral condition.⁸ Although the precise form of the Hadamard series

⁷In fact, Radzikowski's result is more general than that stated here.

⁸We will only need to consider two-point functions; however, a generalisation of the microlocal spectral condition for higher n -point functions has been given by Brunetti *et al.* [12], and one would say that a state is globally Hadamard if all its n -point functions have the appropriate wave front sets. For quasifree states, this more general definition reduces to condition (2.8) on the two-point function.

will not be used in the present paper, it is nonetheless necessary to know that all globally Hadamard two-point functions share a common singular part. This follows from Radzikowski's result [10] that the Feynman propagator $\omega_F = i\omega_2 + \Delta_A$ associated with ω_2 is a distinguished parametrix for the Klein–Gordon operator⁹ in the sense of Duistermaat and Hörmander [20] with wave front set

$$\text{WF}(\omega_F) = O \cup D, \quad (2.9)$$

where¹⁰ $D = \{(x, k; x, -k) \mid (x, k) \in \dot{T}^*(M)\}$ and

$$\begin{aligned} O = \{ & (x, k; x', -k') \in \dot{T}^*(M \times M) \mid x \neq x', (x, k) \sim (x', k'), \\ & \text{and } \pm k \in \overline{V}_+ \text{ if } x \in J^\pm(x') \} . \end{aligned} \quad (2.10)$$

Here, $J^\pm(x')$ denotes the causal future (+) or past (−) of x' . By Theorem 6.5.3 in [20] ω_F is unique up to the addition of a smooth function. Accordingly, the difference between any two globally Hadamard two-point functions is smooth.

2.3 Pseudo-topologies, pull-backs and products

Because the wave front set provides precise local information about the singularity structure of distributions, it has natural applications to the questions of when it is possible to multiply two distributions together, or to pull back a distribution from one manifold to another. The constructions given by Hörmander in [19] extend the usual definitions for smooth functions by continuity with respect to what is now called the Hörmander pseudo-topology, and which we now briefly describe.

First, if X is an open subset of \mathbb{R}^n and Γ is a closed cone¹¹ in $X \times \mathbb{R}^n \setminus \{0\}$ we define ${}'_\Gamma(X)$ to be the set of $u \in {}'(X)$ with $\text{WF}(u) \in \Gamma$. We will say that a sequence $u_r \in {}'_\Gamma(X)$ converges to $u \in {}'_\Gamma(X)$ with respect to the Hörmander pseudo-topology if $u_r \rightarrow u$ in the weak-* sense (i.e., $u_r(f) \rightarrow u(f)$ for each test function f) and the quantities $\sup_{\xi \in V} |\xi|^N |\widehat{\varphi u_r}(\xi)|$ are uniformly bounded in r for each $\varphi \in C_0^\infty(X)$, each closed cone $V \in \mathbb{R}^n$ with $\Gamma \cap (\text{supp } \varphi \times V) = \emptyset$ and each $N = 1, 2, \dots$

If X is now a smooth manifold and Γ is a closed cone in $\dot{T}^*(X)$, we define ${}'_\Gamma(X)$ to be the set of $u \in {}'(X)$ with $\text{WF}(u) \in \Gamma$. A sequence $u_r \in {}'_\Gamma(X)$ is said to converge to $u \in {}'_\Gamma(X)$ with respect to the Hörmander pseudo-topology if for some partition of unity φ_i subordinate to a covering by charts (X_i, κ_i) we have $(\varphi_i u_r) \circ \kappa_i^{-1} \rightarrow (\varphi_i u) \circ \kappa_i^{-1}$ in ${}'_\Gamma(\tilde{X}_i)$ as $r \rightarrow \infty$ for each i . Here, we have denoted $\tilde{X}_i = \kappa_i(X_i)$ and

$$\Gamma_i = \{(\kappa_i(x), \xi) \mid x \in X_i, (x, {}^t\kappa'_i(x)\xi) \in \Gamma\}, \quad (2.11)$$

⁹That is, the action of $\square + m^2$ on either ‘slot’ of the bidistribution ω_F yields a smooth function.

¹⁰See footnote 4 above for purposes of comparison with [12].

¹¹That is, Γ is topologically closed and, for each $x \in X$, the set $\Gamma_x = \{k \mid (x, k) \in \Gamma\}$ is a cone in $\mathbb{R}^n \setminus \{0\}$.

where $N_i \subset X_i$ is some arbitrarily chosen closed neighbourhood of $\text{supp } \varphi_i$. This turns out to be an invariant definition.

In the sequel, we will use the fact that any $u \in {}'_\Gamma(X)$ may be approximated by a regularising sequence of smooth compactly supported functions, as shown by the following extension of Theorem 8.2.3 in [9]. Recall that the convolution $u \star \chi$ of a distribution $u \in {}'_\Gamma(X)$ with $\chi \in C_0^\infty(\mathbb{R}^n)$ is defined to be the (smooth) function $x \mapsto u(\chi(x - \cdot))$, and has support contained in $\text{supp } u + \text{supp } \chi$.

Lemma 2.1 *Let X be a smooth n -dimensional manifold and suppose $u \in {}'_\Gamma(X)$ is of compact support. Choose a partition of unity φ_i subordinate to a covering of charts (X_i, κ_i) such that $\text{supp } u$ intersects only finitely many $\text{supp } \varphi_i$ (say $i = 1, \dots, I$). Let χ_r ($r = 1, 2, \dots$) be any sequence of smooth non-negative functions in $C_0^\infty(\mathbb{R}^n)$ with $\int \chi_r d^n x = 1$ and $\text{supp } \chi_r$ sufficiently small that $\text{supp } \chi_r + \text{supp } \varphi_i \circ \kappa_i^{-1} \subset \tilde{X}_i$ for each $r = 1, 2, \dots$ and $i = 1, \dots, I$. If $\text{supp } \chi_r \rightarrow \{0\}$ as $r \rightarrow \infty$ then the distributions $u_r = \sum_i [(\varphi_i u) \circ \kappa_i^{-1}] \star \chi_r \circ \kappa_i$ converge to u in the Hörmander pseudo-topology on ${}'_\Gamma(X)$.*

Proof: Since the sequence χ_r is an approximate identity in the image of each chart X_i , for $i = 1, \dots, I$, we have $[(\varphi_i u) \circ \kappa_i^{-1}] \star \chi_r \rightarrow (\varphi_i u) \circ \kappa_i^{-1}$ in each ${}'_\Gamma(\tilde{X}_i)$ by Theorem 8.2.3 in [9]. ■

For future reference, let us note that (by using mollifiers of the form $\chi_r \otimes \chi_r$) a distribution $u \otimes v \in {}'_\Gamma(X \times X)$ may be regularised by a sequence $u_r \otimes v_r$ where $u_r, v_r \in C_0^\infty(X)$.

Returning to the construction of pull-backs, suppose X and Y are manifolds and $\varphi : Y \rightarrow X$ is smooth. Given $u \in {}'_\Gamma(X)$, Theorem 2.5.11' in [19] constructs the pull-back $\varphi^* u$ as a distribution on Y provided $\text{WF}(u) \cap N_\varphi = \emptyset$, where

$$N_\varphi = \{(\varphi(y), \xi) \in T^*(X) \mid {}^t\varphi'(y)\xi = 0\} \quad (2.12)$$

defines the set of normals of the map φ . The wave front set of the pull-back is constrained by

$$\text{WF}(\varphi^* u) \subset \varphi^* \text{WF}(u) = \{(y, {}^t\varphi'(y)\xi) \mid (\varphi(y), \xi) \in \text{WF}(u)\}. \quad (2.13)$$

If u is smooth, the pull-back reduces to ordinary composition $\varphi^* u(y) = u(\varphi(y))$; the pull-back operation is also sequentially continuous with respect to the Hörmander pseudo-topology on ${}'_\Gamma(X)$ for any closed cone $\Gamma \subset \dot{T}^*(X)$ having empty intersection with N_φ .

The product of two distributions may be defined—under suitable conditions—using a closely related construction (Theorem 2.5.10 in [19]). Provided Γ_1, Γ_2 are closed cones in $\dot{T}^*(X)$ whose sum, defined by

$$\Gamma_1 + \Gamma_2 = \{(x, \xi_1 + \xi_2) \mid (x, \xi_i) \in \Gamma_i\}, \quad (2.14)$$

has no intersection with the zero section of $T^*(X)$, we may define a product $u_1 u_2$ for any $u_i \in {}'_\Gamma(X)$. The product is sequentially continuous in each factor with respect to the

Hörmander pseudo-topologies on the $\dot{L}'_{\Gamma_i}(X)$, and agrees with the usual product if both the u_i are smooth.

These constructions are related by the formula

$$\varphi^*u(g) = (g_\varphi u)(1_X) \quad \forall g \in C_0^\infty(Y), \quad (2.15)$$

where X, Y and φ are as above, 1_X is the unit function on X and $g_\varphi \in \dot{L}'(Y)$ is the compactly supported distribution

$$g_\varphi(f) = \int_Y \sigma_Y(y) g(y) f(\varphi(y)) dy \quad f \in C_0^\infty(X). \quad (2.16)$$

The wave front set of g_φ is easily seen to lie within \dot{N}_φ , so the pull-back φ^*u exists if and only if the product $g_\varphi u$ does. One could in fact adopt Eq. (2.15) as the definition of the pull-back; it can also be proved directly from the analogous statement for smooth u , using sequential continuity of both the pull-back and the product.

This relationship underlies the following result.

Theorem 2.2 *Let X and Y be smooth manifolds equipped with smooth positive densities σ_X and σ_Y , and suppose $\gamma : Y \rightarrow X$ is smooth. If $u \in \dot{L}'(X \times X)$ is of positive type and $\text{WF}(u) \cap N_\varphi = \emptyset$, where $\varphi(y, y') = (\gamma(y), \gamma(y'))$, then φ^*u is of positive type.*

Proof: Each $g \in C_0^\infty(Y)$ defines a compactly supported distribution g_γ by (2.16) above, with φ replaced by γ . Moreover, $\overline{g_\gamma} \otimes g_\gamma = (\overline{g} \otimes g)_\varphi$, so its wave front set lies in \dot{N}_φ . Choose any closed cone $\Gamma \in \dot{T}^*(X \times X)$ with $\dot{N}_\varphi \in \Gamma$ and $\text{WF}(u) \cap \Gamma = \emptyset$, and (cf. the remark following Lemma 2.1) pick a regularising sequence $\overline{g^{(r)}} \otimes g^{(r)}$ converging to $\overline{g_\gamma} \otimes g_\gamma$ in $\dot{L}'_{\Gamma}(X \times X)$ with $g^{(r)} \in C_0^\infty(X)$ for each r . To prove the result, we use sequential continuity of the product and the calculation

$$\begin{aligned} \varphi^*u(g \otimes g) &= \lim_{r \rightarrow \infty} \left((\overline{g_\gamma^{(r)}} \otimes g_\gamma^{(r)}) u \right) (1_{X \times X}) \\ &= \lim_{r \rightarrow \infty} u \left(\overline{g_\gamma^{(r)}} \otimes g_\gamma^{(r)} \right) \\ &\geq 0 \end{aligned} \quad (2.17)$$

to conclude that φ^*u is of positive type. ■

3 The point-split energy density

After these lengthy preliminaries, the proof of the general quantum inequality is quite straightforward. We begin by defining our point-split energy density, and deriving some of its properties.

The classical stress-energy tensor associated with the Klein-Gordon equation (2.1) on (M, \mathbf{g}) is

$$T_{ab} = \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} g^{cd} \nabla_c \phi \nabla_d \phi + \frac{1}{2} m^2 \phi^2. \quad (3.1)$$

Let γ be a smooth timelike curve in M , parametrised by its proper time, and let Γ be a tubular neighbourhood of γ . We may choose a smooth orthonormal frame $\{v_\mu^a \mid \mu = 0, \dots, N-1\}$ in Γ so that $g^{ab} = \eta^{\mu\nu} v_\mu^a v_\nu^b$ and the restriction of v_0^a to γ equals the four-velocity $\dot{\gamma}^a(\tau)$ of the trajectory, also denoted $u^a(\tau)$.

Now, an observer moving along γ measures energy density $T(\tau) = u^a(\tau) u^b(\tau) T_{ab}(\gamma(\tau))$, which may be written

$$T(\tau) = \frac{1}{2} \left(\sum_{\mu=0}^{N-1} v_\mu^a v_\mu^b \right) \nabla_a \phi \nabla_b \phi + \frac{1}{2} m^2 \phi^2 \quad (3.2)$$

in terms of the frame. This quantity is clearly the restriction to the diagonal $\tau = \tau'$ of the smooth bi-scalar field

$$T(\tau, \tau') = \frac{1}{2} \left(\sum_{\mu=0}^{N-1} v_\mu^a(\tau) v_\mu^b(\tau') \right) \nabla_a \phi|_{\gamma(\tau)} \nabla_b \phi|_{\gamma(\tau')} + \frac{1}{2} m^2 \phi(\gamma(\tau)) \phi(\gamma(\tau')), \quad (3.3)$$

which we will call the (classical) point-split energy density. Of course, in contrast to $T(\tau)$, it depends on the frame – a point which should be borne in mind below.

The quantised version of $T(\tau, \tau')$ is easily constructed: given a state ω on $\mathfrak{A}(M, \mathbf{g})$ whose two-point function ω_2 obeys the microlocal spectral condition (2.8), we define the distribution $\langle T \rangle_\omega$ on \mathbb{R}^2 by

$$\langle T \rangle_\omega = \frac{1}{2} \sum_{\mu=0}^{N-1} \varphi^* \left((v_\mu^a \nabla_a \otimes v_\mu^b \nabla_b) \omega_2 \right) + \frac{1}{2} m^2 \varphi^* \omega_2, \quad (3.4)$$

where φ^* is the pull-back from $M \times M$ to \mathbb{R}^2 induced by the map $\varphi(\tau, \tau') = (\gamma(\tau), \gamma(\tau'))$. Noting that ${}^t\varphi'(\tau, \tau') : T_{(\gamma(\tau), \gamma(\tau'))}^*(M \times M) \rightarrow \mathbb{R}^2$ is the linear map

$${}^t\varphi'(\tau, \tau') : (k, k') \mapsto (u^a(\tau) k_a, u^{b'}(\tau') k_{b'}) \quad (3.5)$$

we see that φ has the following set of normals:

$$N_\varphi = \left\{ (\gamma(\tau), k; \gamma(\tau'), k') \mid k_a u^a(\tau) = k_{b'} u^{b'}(\tau') = 0 \right\}. \quad (3.6)$$

To check that $\langle T \rangle_\omega$ is well-defined let us first consider the term $\varphi^* \omega_2$. Suppose $(x, k; x', k') \in \text{WF}(\omega_2) \cap N_\varphi$. Then $x = \gamma(\tau)$ and $x' = \gamma(\tau')$ for some τ, τ' ; furthermore k and k' are required to be both null (so as to be in $\text{WF}(\omega_2)$) and to annihilate the timelike vectors $u^a(\tau)$ and $u^{b'}(\tau')$ (so as to be in N_φ). But no non-zero null covector can annihilate

a non-zero timelike vector. Hence $\text{WF}(\omega_2) \cap N_\varphi$ is empty and the pull-back $\varphi^*\omega_2$ is well-defined, with wave front set contained in $\varphi^*\text{WF}(\omega_2)$.

Now, referring to Eqs. (2.8) and (2.13), we have $(\tau, \zeta; \tau', -\zeta') \in \varphi^*\text{WF}(\omega_2)$ if and only if

$$(\zeta, -\zeta') = ({}^t\varphi'(\tau, \tau'))(k, -k') = (u^a(\tau)k_a, -u^{b'}(\tau')k'_{b'}) \quad (3.7)$$

for some k, k' such that $(\gamma(\tau), k) \sim (\gamma(\tau'), k')$ and $k \in \overline{V}^+$. In particular, $\varphi^*\text{WF}(\omega_2)$ contains all points of the form $(\tau, \zeta; \tau', -\zeta')$ with $\zeta > 0$; however, if (M, \mathbf{g}) has compact Cauchy surfaces one will generally find distinct null-related points of γ and therefore an increased wave front set. Nonetheless, because the vectors $u^a(\tau)$, $u^a(\tau')$ and the covectors k_a , k'_a are all future pointing we have $\zeta, \zeta' > 0$ in all cases, so

$$\text{WF}(\varphi^*\omega_2) \subset \varphi^*\text{WF}(\omega_2) \subset \{(\tau, \zeta; \tau', -\zeta') \mid \zeta, \zeta' > 0\}, \quad (3.8)$$

and this result will be enough for our purposes. We also note that $\varphi^*\omega_2$ is of positive type by Theorem 2.2.

Coupled with the non-expansion of the wave front set under partial differential operators, the same reasoning shows that the other terms in (3.4) are also well-defined with wave front sets contained in the set on the right-hand side of the previous expression. In addition, they are of positive type by Theorem 2.2 since

$$\left((v_\mu^a \nabla_a \otimes v_\mu^{b'} \nabla_{b'}) \omega_2 \right) (\bar{f} \otimes f) = \omega_2(\overline{\nabla_a(v_\mu^a f)} \otimes \nabla_{b'}(v_\mu^{b'} f)) \geq 0 \quad (3.9)$$

for each $f \in C_0^\infty(M)$.

To summarise, $\langle T \rangle_\omega$ is a well-defined distribution on \mathbb{R}^2 , which is of positive type and has wave front set obeying

$$\text{WF}(\langle T \rangle_\omega) \subset \{(\tau, \zeta; \tau', -\zeta') \mid \zeta, \zeta' > 0\}. \quad (3.10)$$

Since we have $\text{WF}(f\langle T \rangle_\omega) \subset \text{WF}(\langle T \rangle_\omega)$ for any $f \in C_0^\infty(\mathbb{R}^2)$, Proposition 8.1.3 in [9] entails that $\Sigma(f\langle T \rangle_\omega) \subset \{(\zeta, -\zeta') \mid \zeta, \zeta' > 0\}$. In particular, the Fourier transform $[f\langle T \rangle_\omega]^\wedge(-\alpha, \alpha)$ decays rapidly as $\alpha \rightarrow +\infty$.

Finally, we note that $\langle T \rangle_\omega$ depends on the restriction of the frame v_μ^a to γ , but not on its values on $\Gamma \setminus \gamma$.

4 Main result

We are now in a position to state and prove our main result. As above, γ is assumed to be a smooth timelike curve parametrised by proper time in a globally hyperbolic spacetime (M, \mathbf{g}) of dimension $N \geq 2$ (with, strictly, the requirement $m > 0$ if $N = 2$). For a state with a globally Hadamard two-point function, the bi-distribution $\langle T \rangle_\omega$ is defined as in Sect. 3 using a fixed choice of orthonormal frame on (a tubular neighbourhood of) γ .

Theorem 4.1 *Let ω and ω_0 be states on $\mathfrak{A}(M, \mathbf{g})$ with globally Hadamard two-point functions and define the normal ordered energy density relative to ω_0 by $\langle : T : \rangle_\omega = \langle T \rangle_\omega - \langle T \rangle_{\omega_0}$. Then $\langle : T : \rangle_\omega$ is smooth, and the quantum inequality*

$$\int d\tau |g(\tau)|^2 \langle : T : \rangle_\omega(\tau, \tau) \geq - \int_0^\infty \frac{d\alpha}{\pi} [(\bar{g} \otimes g) \langle T \rangle_{\omega_0}]^\wedge(-\alpha, \alpha) \quad (4.1)$$

holds for all $g \in C_0^\infty(\mathbb{R})$ (and the right-hand side of (4.1) is convergent for all such g).

The fact that the right-hand side converges is, of course, necessary for the inequality to be other than vacuous. We remark that the bound presumably depends on the choice of frame along γ . No assertion is made that this quantum inequality is the best possible.

Proof: First, we note that $\langle : T : \rangle_\omega$ is real-valued and symmetric in τ, τ' because the two-point functions of ω and ω_0 have identical anti-symmetric parts and real symmetric parts. Moreover, the difference of the two-point functions is smooth by the Hadamard condition, so $\langle : T : \rangle_\omega$ is smooth.

Now, because $\langle T \rangle_\omega$ is of positive type, we have

$$\begin{aligned} \int d\tau d\tau' \overline{g(\tau)} g(\tau') \langle : T : \rangle_\omega(\tau, \tau') &= \langle : T : \rangle_\omega(\bar{g} \otimes g) \\ &\geq -\langle T \rangle_{\omega_0}(\bar{g} \otimes g) \\ &= -[(\bar{g} \otimes g) \langle T \rangle_{\omega_0}]^\wedge(0, 0) \end{aligned} \quad (4.2)$$

for all test functions $g \in C_0^\infty(\mathbb{R})$. Since the integrand on the left-hand side is smooth and compactly supported we may write

$$\begin{aligned} \int d\tau |g(\tau)|^2 \langle : T : \rangle_\omega(\tau, \tau) &= \int_{-\infty}^\infty \frac{d\alpha}{2\pi} \int d\tau d\tau' \overline{g(\tau)} g(\tau') e^{-i\alpha(\tau-\tau')} \langle : T : \rangle_\omega(\tau, \tau') \\ &= \int_0^\infty \frac{d\alpha}{\pi} \int d\tau d\tau' \overline{g(\tau)} g(\tau') e^{-i\alpha(\tau-\tau')} \langle : T : \rangle_\omega(\tau, \tau'), \end{aligned} \quad (4.3)$$

where the final step uses the fact that $\langle : T : \rangle_\omega$ is real and symmetric. Applying inequality (4.2) to the test function $g_\alpha(t) = g(t)e^{i\alpha t}$, we note that the integrand in the final line of (4.3) is greater than or equal to $-[(\bar{g} \otimes g) \langle T \rangle_{\omega_0}]^\wedge(-\alpha, \alpha)$. But this decays rapidly as $\alpha \rightarrow \infty$ by the comments following Eq. (3.10) and is therefore integrable, which proves the result. ■

5 Example: stationary spacetimes

We now derive a more compact form of our general quantum inequality for the case of stationary spacetimes admitting a Hadamard stationary ground state ω_0 on $\mathfrak{A}(M, \mathbf{g})$, which we will adopt as our reference state.

Recall that (M, \mathbf{g}) is said to be stationary if there is a 1-parameter group of isometries ψ_t , all of whose orbits are timelike, generated by a Killing vector field ξ^a . A state ω_0 is said to be stationary if $\omega_0(\alpha_t A) = \omega_0(A)$ for each $A \in \mathfrak{A}(M, \mathbf{g})$ and all $t \in \mathbb{R}$, where α_t denotes the automorphism of $\mathfrak{A}(M, \mathbf{g})$ defined by

$$\alpha_t \phi(f) = \phi(\psi_t^* f) \quad \forall t \in \mathbb{R}, f \in C_0^\infty(M), \quad (5.1)$$

where $\psi_t^* f$ is the pull-back of f by ψ_t , i.e., $\psi_t^* f(p) = f(\psi_t p)$ for $p \in M$. The state is a ground state if, in addition, these automorphisms are implemented in the GNS representation¹² (ρ, \cdot, Ω) of ω_0 by

$$\rho(\alpha_t A) = e^{iHt} \rho(A) e^{-iHt} \quad \forall t \in \mathbb{R}, A \in \mathfrak{A}(M, \mathbf{g}), \quad (5.2)$$

where H is a positive self-adjoint operator on \cdot . The existence of a Hadamard stationary ground state is a nontrivial restriction on (M, \mathbf{g}) —for example, there is no stationary Hadamard state on the Kerr spacetime [18]—but such a state will certainly exist if $m > 0$ and, for some $\epsilon > 0$ and Cauchy surface $\Sigma \subset M$, we have $\xi^a \xi_a \geq \epsilon \xi^a n_a \geq \epsilon^2$ on Σ , where n^a is the unit normal to Σ . See, e.g., §4.3 in [21].

Now consider an stationary observer, whose trajectory γ is therefore an orbit of ψ_t , $\gamma(t) = \psi_t p_0$ for some fixed p_0 . Since $\xi^a \xi_a$ is constant on such orbits, we may assume without loss of generality that $\xi^a \xi_a = 1$ on γ and so this is a proper time parametrisation. Fix any open neighbourhood \mathcal{O} of p_0 in a Cauchy surface through p_0 sufficiently small that a smooth orthonormal frame v_μ^a for \mathbf{g} may be chosen in \mathcal{O} with v_0^a parallel to ξ^a , and use the action of ψ_t to extend this framing to the tubular neighbourhood $\Gamma = \{\psi_t p \mid p \in \mathcal{O}, t \in \mathbb{R}\}$ of γ so that

$$v_\mu^a(\psi_t p) = \psi_{t*} v_\mu^a(p) \quad (5.3)$$

for each $p \in \mathcal{O}$. The procedure is well-defined because it entails that the Lie derivative $\mathcal{L}_\xi v_\mu^a$ vanishes for each μ , so both sides of the equation $g^{ab} = \eta^{\mu\nu} v_\mu^a v_\nu^b$ are preserved under the flow.

We use this framing to construct $\langle T \rangle_{\omega_0}$ by the method of Sect. 3. This distribution is invariant with respect to the translation $(\tau, \tau') \mapsto (\tau + t, \tau' + t)$ for any t as a consequence of stationarity, and this combined with the fact that ω_0 is a ground state leads to the following statement, which is proved in the Appendix.

Proposition 5.1 *There exists a tempered distribution \mathcal{T} such that*

$$\langle T \rangle_{\omega_0}(f \otimes g) = \mathcal{T}(f \star \tilde{g}), \quad (5.4)$$

where $\tilde{g}(\tau) = g(-\tau)$. Furthermore, the Fourier transform of \mathcal{T} is a positive measure of at most polynomial growth with support contained in \mathbb{R}^+ .

¹²Given a state ω on a unital $*$ -algebra \mathfrak{A} , the GNS representation consists of a Hilbert space \cdot , a representation ρ of \mathfrak{A} as (unbounded) operators on a common dense domain \cdot in \cdot , and a distinguished unit vector $\Omega \in \cdot$ such that $\omega(A) = \langle \Omega \mid \rho(A) \Omega \rangle$ for all $A \in \mathfrak{A}$. The vector Ω is also cyclic, in the sense that $\{\rho(A) \Omega \mid A \in \mathfrak{A}\}$ is dense in \cdot .

Noting that $2\pi\mathcal{T}(\bar{g} \star \tilde{g}) = \widehat{\mathcal{T}(\bar{g} \star \tilde{g})} = \widehat{\mathcal{T}}(|\hat{g}|^2)$ we have

$$\langle T \rangle_{\omega_0}(\bar{g} \otimes g) = \int_0^\infty \frac{d\zeta}{2\pi} \widehat{\mathcal{T}}(\zeta) |\hat{g}(\zeta)|^2 \quad (5.5)$$

for any $g \in C_0^\infty(\mathbb{R})$. Accordingly, the quantum inequality becomes

$$\begin{aligned} \int d\tau |g(\tau)|^2 \langle : T : \rangle_\omega(\tau, \tau) &\geq - \int_0^\infty \frac{d\alpha}{\pi} \int_0^\infty \frac{d\zeta}{2\pi} \widehat{\mathcal{T}}(\zeta) |\hat{g}(\zeta + \alpha)|^2 \\ &= - \frac{1}{\pi} \int_0^\infty du |\hat{g}(u)|^2 Q(u), \end{aligned} \quad (5.6)$$

where the (positive) polynomially bounded function Q is given on \mathbb{R}^+ by

$$Q(u) = \int_0^u \frac{d\zeta}{2\pi} \widehat{\mathcal{T}}(\zeta). \quad (5.7)$$

As described in the introduction, this inequality generalises that given for the static case in [5].

One may think of $Q(u)$ as a cut-off value of the infinite quantity “ $\mathcal{T}(0)$ ”—the coincidence limit of the unrenormalised point-split energy density—which is formally equal to $Q(\infty)$. The rate of divergence of $Q(u)$ is fixed by dimensional considerations to be $O(u^N)$ where N is the spacetime dimension.

Finally, we reformulate (5.6) as a positivity condition on a pseudodifferential operator. Noting that the left-hand side of this inequality is invariant under $g \mapsto \bar{g}$, and writing $\rho_\omega(\tau) = \langle : T : \rangle_\omega(\tau, \tau)$, we have

$$\begin{aligned} \int d\tau |g(\tau)|^2 \rho_\omega(\tau) &\geq - \frac{1}{2\pi} du \int_0^\infty \left(|\hat{g}(u)|^2 + |\widehat{\bar{g}}(u)|^2 \right) Q(u) \\ &= - \frac{1}{2\pi} \int_{-\infty}^\infty du |\hat{g}(u)|^2 Q(|u|), \end{aligned} \quad (5.8)$$

using $\widehat{\bar{g}}(u) = \hat{g}(-u)$. But by the spectral theorem, the final expression is just the matrix element $\langle g | Q(|iD|)g \rangle$, where $\langle \cdot | \cdot \rangle$ is the usual inner product on $L^2(\mathbb{R})$, and iD is the self-adjoint operator $(iDg)(\tau) = ig'(\tau)$ with domain equal to the Sobolev space $W^{1,2}(\mathbb{R})$ [22]. Thus (5.8) asserts that the symmetric operator $Q(|iD|) + \rho_\omega$ (where ρ_ω acts by multiplication) is positive on $C_0^\infty(\mathbb{R})$. Under circumstances in which $C_0^\infty(\mathbb{R})$ is a form core for this operator (e.g., if ρ_ω is bounded) positivity on $C_0^\infty(\mathbb{R})$ is equivalent to the positivity of the self-adjoint operator $H_\omega = Q(|iD|) + \dot{\rho}_\omega$ where the dot denotes the sum in the sense of quadratic forms. (Since ρ_ω is smooth, this reduces to the ordinary operator sum if ρ_ω is also bounded.) Under certain conditions, positivity of H_ω will in turn be equivalent to the absence of negative eigenvalues. In these circumstances, one can determine whether or not a given candidate energy density ρ is compatible with the quantum inequalities by considering the corresponding eigenvalue problem. This viewpoint has been explored recently in [14] in the case of $2m$ -dimensional Minkowski

space, where $Q(|iD|)$ is equal to $(-1)^m D^{2m}$ (up to constant factors). If a negative eigenvalue is found, then we have shown that the candidate ρ is not the energy density derived from any globally Hadamard state on (M, \mathbf{g}) . It would be interesting to gain further insight into the conditions under which the converse holds.

6 Conclusion

We have described a new and general worldline quantum inequality, which is both rigorous and explicit. As mentioned above, the results given here reduce to those of [4, 5] in the static case; indeed, one may regard the present derivation as the correct setting for that earlier work. Various future directions are possible. First, it would be interesting to examine the asymptotic behaviour of our bound both for sampling functions of very short duration (in which case one expects to find a ‘universal’ leading order term depending only on the short-distance structure of the Hadamard form), and of very long duration (to investigate what averaged weak energy results are possible). Second, we have observed that our results are not the best possible worldline inequalities, and this raises the question of whether it is possible to tune the general argument given here to produce sharper bounds. Finally, we expect that our general approach can also be adapted to provide quantum inequalities for spacetime averages of the stress-energy tensor, thus giving an alternative method to that of Helfer [6]. We hope to return to these issues elsewhere.

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A Proof of Proposition 5.1

Without loss of generality, we may introduce stationary coordinates $(x_0, x_1, \dots, x_{N-1})$ on Γ so that $\gamma(\tau)$ has coordinates $(\tau, 0, \dots, 0)$. Fix $\chi \in C_0^\infty(\mathbb{R}^N)$ with $\int \chi(\underline{x}) d^N \underline{x} = 1$ and set $\chi_r(\underline{x}) = \chi(\underline{x}/r)$ for $r = 1, 2, \dots$. We will adopt the convention that given any test function $f \in C_0^\infty(\mathbb{R})$, $f^{(r)}$ will denote the regularisation of f_γ (cf. Eq. (2.16)) obtained by convolution with χ_r . This regularisation is covariant with respect to the stationary isometry group in the sense that

$$(\psi_\tau^* f)^{(r)} = \psi_\tau^* f^{(r)}, \quad (\text{A.1})$$

where, for notational convenience, we use ψ_τ to denote both the stationary isometry of (M, \mathbf{g}) and the translation $t \mapsto t + \tau$ on \mathbb{R} .

Next, define $\langle T \rangle_{\omega_0}^{(r)} \in '(\mathbb{R}^2)$ by

$$\langle T \rangle_{\omega_0}^{(r)}(f \otimes g) = \frac{1}{2} \sum_{\mu=0}^{N-1} \left((v_\mu^a \nabla_a \otimes v_\mu^{b'} \nabla_{b'}) \omega_2 \right) (f^{(r)} \otimes g^{(r)}) + \frac{1}{2} m^2 \omega_2(f^{(r)} \otimes g^{(r)}). \quad (\text{A.2})$$

The covariance relation (A.1) and the stationarity of ω_0 imply immediately that each $\langle T \rangle_{\omega_0}^{(r)}$ is translationally invariant under the translation $(t, t') \mapsto (t + \tau, t' + \tau)$ for each τ . Since $f^{(r)} \otimes g^{(r)}$ is a regularising sequence for $f_\gamma \otimes g_\gamma = (f \otimes g)_\varphi$ we have $\langle T \rangle_{\omega_0}^{(r)}(f \otimes g) \rightarrow \langle T \rangle_{\omega_0}(f \otimes g)$ as $r \rightarrow \infty$ for each $f, g \in C_0^\infty(\mathbb{R})$, and deduce that $\langle T \rangle_{\omega_0}$ is also translationally invariant.

It is now standard that there exists $\mathcal{T} \in \mathcal{S}'(\mathbb{R})$ such that

$$\langle T \rangle_{\omega_0}(f \otimes g) = \mathcal{T}(f \star \tilde{g}), \quad (\text{A.3})$$

where $\tilde{g}(\tau) = g(-\tau)$. Since $\langle T \rangle_{\omega_0}$ is positive type in the sense defined in Sect. 2, the distribution \mathcal{T} is of positive type in the sense that $\mathcal{T}(\bar{f} \star \tilde{f}) \geq 0$ for all $f \in C_0^\infty(\mathbb{R})$. The Bochner-Schwartz theorem (Theorem IX.10 in [23]) implies that \mathcal{T} is in fact a tempered distribution whose Fourier transform $\widehat{\mathcal{T}}$ is a polynomially bounded positive measure.

It remains to show that $\text{supp } \widehat{\mathcal{T}} \subset \mathbb{R}^+$. To see this, note that we have tempered distributions $\mathcal{T}^{(r)}$ such that $\langle T \rangle_{\omega_0}^{(r)}(f \otimes g) = \mathcal{T}^{(r)}(f \star \tilde{g})$ whose Fourier transforms are also polynomially bounded positive measures. These Fourier transforms are supported in \mathbb{R}^+ , as may be seen from the argument of Theorem IX.32 in [23] (but applied only to time translations, rather the Poincaré group) and using the fact that the isometry group is represented by e^{iHt} with positive H in the GNS representation of the ground state ω_0 . Since the $\mathcal{T}^{(r)}$ converge in the weak-* sense to \mathcal{T} , we have $\text{supp } \widehat{\mathcal{T}} \subset \mathbb{R}^+$ as required.

References

- [1] L.H. Ford and T.A. Roman, ‘Restrictions on negative energy density in flat space-time’ Phys. Rev. D **55**, 2082 (1997)
- [2] M.J. Pfenning and L.H. Ford, ‘Scalar field quantum inequalities in static space-times’ Phys. Rev. D **57**, 3489 (1998)
- [3] É.É. Flanagan, ‘Quantum inequalities in two-dimensional Minkowski spacetime’ Phys. Rev. D **56**, 4922 (1997)
- [4] C.J. Fewster and S.P. Eveson, ‘Bounds on negative energy densities in flat space-time’ Phys. Rev. D **58**, 084010 (1998)
- [5] C.J. Fewster and E. Teo, ‘Bounds on negative energy densities in static space-times’ Phys. Rev. D **59**, 104016 (1999) [gr-qc/9812032](#)
- [6] A.D. Helfer, ‘The Hamiltonians of linear quantum fields II: Classically positive Hamiltonians’, [hep-th/9908012](#)
- [7] L.H. Ford, ‘Quantum coherence effects and the second law of thermodynamics’ Proc. R. Soc. Lond. A **364**, 227 (1978)

- [8] A.S. Wightman, ‘Introduction to some aspects of the relativistic dynamics of quantized fields’ in *High energy electromagnetic interactions and field theory* Cargèse lectures in theoretical physics, Ed. M. Lévy (Gordon and Breach, New York, 1967)
- [9] L. Hörmander, *The analysis of linear partial differential operators I: Distribution theory and Fourier analysis* 2nd ed. (Springer-Verlag, Berlin, 1990)
- [10] M.J. Radzikowski, ‘Micro-local approach to the Hadamard condition in quantum field theory on curved space-time’, *Commun. Math. Phys.* **179** (1996) 529–553
- [11] M. Köhler, ‘New examples for Wightman fields on a manifold’ *Class. Quantum Grav.* **12** (1995) 1413–1427
- [12] R. Brunetti, K. Fredenhagen and M. Köhler, ‘The microlocal spectrum condition and Wick polynomials of free fields in curved spacetimes’ *Commun. Math. Phys.* **180** (1996) 633–652
- [13] S.A. Fulling, F.J. Narcowich and R.M. Wald, ‘Singularity structure of the two-point function in quantum field theory in curved spacetime, II’ *Ann. Phys.* **136** (1981) 243–272
- [14] C.J. Fewster and E. Teo, ‘Quantum inequalities and “quantum interest” as eigenvalue problems’ [gr-qc/9908073](#)
- [15] L.H. Ford and T.A. Roman, ‘The quantum interest conjecture’ [gr-qc/9901074](#)
- [16] B.S. Kay, ‘The principle of locality and quantum field theory on (non globally hyperbolic) curved spacetimes’ *Rev. Math. Phys. Special Issue* (1992) 167–195
- [17] R.M. Wald, *General Relativity* (University of Chicago, Chicago, 1984)
- [18] B.S. Kay and R.M. Wald, ‘Theorems on the uniqueness and thermal properties of stationary, nonsingular, quasifree states on spacetimes with a bifurcate Killing horizon’ *Phys. Rep.* **207** (1991) 49–136
- [19] L. Hörmander, ‘Fourier integral operators. I’ *Acta Math.* **127** (1971) 79–183
- [20] J.J. Duistermaat and L. Hörmander, ‘Fourier integral operators. II’ *Acta Math.* **128** (1972) 183–269
- [21] R.M. Wald, *Quantum field theory in curved spacetime and black hole thermodynamics* (University of Chicago, Chicago, 1994)
- [22] R.A. Adams, *Sobolev spaces* (Academic Press, New York, 1975)
- [23] M. Reed and B. Simon, *Methods of modern mathematical physics II: Fourier analysis, self-adjointness* (Academic Press, San Diego, 1975)